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The Clebsch-Gordan problem for quiver representations

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Quivers. A quiver Q is a directed graph consisting of a set of vertices Q_0 and a set of arrows Q_1. Below are some examples of quivers:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \qquad \bullet \circlearrowleft \qquad \bullet \rightrightarrows \bullet$$

Representations. A representation V of a quiver Q consists of a collection of vector spaces V_x, where $x \in Q_0$ and linear maps $V(\alpha) : V_x \to V_y$, where $x \xrightarrow{\alpha} y \in Q_1$. Given representations V and W we define their direct sum $V \oplus W$ and tensor product $V \otimes W$ by

$$(V \oplus W)_x = V_x \oplus W_x, \qquad (V \oplus W)(\alpha) = V(\alpha) \oplus W(\alpha),$$
$$(V \otimes W)_x = V_x \otimes W_x, \qquad (V \otimes W)(\alpha) = V(\alpha) \otimes W(\alpha).$$

Clebsch-Gordan problem. Find the decomposition

$$V \otimes W = \bigoplus_i U_i$$

into indecomposable representations U_i for each pair of indecomposable representations V, W.

Known solutions. For the loop quiver

$$Q : \bullet \circlearrowleft$$

the Clebsch-Gordan problem was solved by Aitken (1935) over an algebraically closed ground field of characteristic zero. For algebraically closed ground fields of positive characteristic the solution was found by Iima-Iwamatsu (2006). Finally, over any perfect ground field the solution was found by Darpö-H (2008).

For Dynkin quivers the solution is known for type $\mathbb{A}$, $\mathbb{D}$, and $\mathbb{E}_6$. The Clebsch-Gordan problem has also been solved for a fairly large class of tame algebas called string algebras.

String algebras. Let I be an ideal in the path algebra $\Bbbk Q$ generated by a set of paths such that the quotient algebra $\Lambda = \Bbbk Q/I$ is finite dimensional. Then Λ is called a string algebra if

1. For each vertex $x \in Q_0$ there are at most two arrows starting (respectively ending) at x.
2. For each arrow $\alpha \in Q_1$ there is at most one $\beta \in Q_1$ such that $\alpha\beta \notin I$ and at most one $\gamma \in Q_1$ such that $\gamma\alpha \notin I$.

Strings and bands. For a quiver morphism $F : P \to Q$ we call (F, P) a shape over Q if for distinct arrows $x \xrightarrow{\alpha} y$ and $x' \xrightarrow{\alpha'} y'$ in P, $F\alpha = F\alpha'$ implies $x \neq x'$ and $y \neq y'$. With each shape we associate two functors

$$\operatorname{rep}_\Bbbk P \underset{F^*}{\overset{F_*}{\rightleftarrows}} \operatorname{rep}_\Bbbk Q.$$

A shape $\mathbf{F} = (F, L)$ is called linear if the underlying graph of L is Dynkin of type $\mathbb{A}$. Let V be the representation

$$\Bbbk \overset{1}{\text{——}} \cdots \overset{1}{\text{——}} \Bbbk$$

and set $S_\mathbf{F} = F_* V$. The representations $S_\mathbf{F}$ are called strings.

A shape $\mathbf{G} = (G, Z)$ is called cyclic if it has trivial automorphism group and the underlying graph of Z is extended Dynkin of type $\tilde{\mathbb{A}}$. Let M be a $\Bbbk[T, T^{-1}]$-module and $\gamma \in Z_1$. Furthermore, let W be the representation

$$\begin{array}{ccccc} & & M & & \\ & / & & \backslash & \\ M & \text{——} & \cdots & \text{——} & M \end{array}$$

where all arrows act as identity except γ which acts by multiplication with T. Set $B_\mathbf{G}(M, \gamma) = G_* W$. The representations $B_\mathbf{G}(M, \gamma)$ are called bands.

It is a classical result due to several authors including Gelfand-Ponomarev and Ringel-Butler that for string algebras, the indecomposable representations are classified by strings and bands. Let $\mathcal{L}(\mathbf{F}, \mathbf{F}')$ be the set of linear connected components of the fibre product $\mathbf{F} \times_Q \mathbf{F}'$. An example of the fibre product of two linear shapes can be found below.

Theorem. *For linear shapes* $\mathbf{F}$, $\mathbf{F}'$, *non-isomorphic cyclic shapes* $\mathbf{G}$, $\mathbf{G}'$, *and* $\Bbbk[T, T^{-1}]$*-modules* M, M' *the following formulae hold:*

$$S_\mathbf{F} \otimes S_{\mathbf{F}'} \xrightarrow{\sim} \bigoplus_{\mathbf{H} \in \mathcal{L}(\mathbf{F}, \mathbf{F}')} S_\mathbf{H}$$

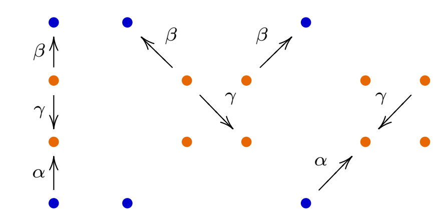
$$S_\mathbf{F} \otimes B_\mathbf{G}(M, \gamma) \xrightarrow{\sim} \bigoplus_{\mathbf{H} \in \mathcal{L}(\mathbf{F}, \mathbf{G})} \dim M S_\mathbf{H}$$

$$B_\mathbf{G}(M, \gamma) \otimes B_{\mathbf{G}'}(M', \gamma') \xrightarrow{\sim} \bigoplus_{\mathbf{H} \in \mathcal{L}(\mathbf{G}, \mathbf{G}')} \dim M \dim M' S_\mathbf{H}$$

$$B_\mathbf{G}(M, \gamma) \otimes B_\mathbf{G}(M', \gamma) \xrightarrow{\sim} B_\mathbf{G}(M \otimes_\Bbbk N, \gamma) \oplus \bigoplus_{\mathbf{H} \in \mathcal{L}(\mathbf{G}, \mathbf{G})} \dim M \dim M' S_\mathbf{H}$$

Example. $Q : \bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet \circlearrowleft \gamma \quad I = \langle \alpha\beta, \gamma^2, (\beta\gamma\alpha)^n \rangle$

$\mathbf{F}$:



$$\mathbf{F}' : \bullet \xleftarrow{\beta} \bullet \xrightarrow{\gamma} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\alpha} \bullet \xleftarrow{\gamma} \bullet$$